THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.; E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN, AND BOMBAY.

VOL. III. DECEMBER, 1904. No. 48.

NOTE ON THE "METHOD OF THE ARITHMETIC MEAN" AS APPLIED TO RATES OF INCREASE.

IT is known to all students of physics that when a number of different observations have been made of the same constant quantity, the best result is, in general, obtained by taking the arithmetic mean of the observed values.

If the observed quantity instead of being constant is increasing at a uniform rate, and we require to determine this rate of increase, we should naturally expect that the proper course to follow would be to find the rates of increase of the quantity in successive equal intervals of time and take their arithmetic mean.

Suppose, then, that $y_0, y_1, y_2, \ldots y_n$ are the observed values of the quantity, the interval between every two consecutive observations being equal to i, and there being altogether n intervals. The rates of increase in the several intervals are

$$\frac{y_1-y_0}{i}, \quad \frac{y_2-y_1}{i}, \ldots \frac{y_n-y_{n-1}}{i}.$$

The arithmetical mean of these is

$$\frac{y_n-y_0}{ni},$$

and thus depends only on the initial and final values and not on the intermediate ones. It is, however, quite contrary to all ideas of common sense that this result should be regarded as giving the most probable value, since its accuracy depends wholly on two measurements and no weight is attached to the intermediate measurements.

If we plot the observed values on squared paper the above result would mean that the rate of increase was determined by the straight line joining the first and last points plotted. It might, however, be found that the intermediate points were very nearly in a straight line, and that, owing to considerable errors in the first and last values, the first and last points were far from being in a line with the rest. It is obvious that in such cases the method would give a bad result.

This difficulty can be overcome by applying the method of Least Squares in the following manner. Let it be assumed that the correct value of the

observed quantity y at time t is given by the formula

$$y = a + bt,$$

where a, b are constants, b denoting the required probable rate of increase. Then by the method of Least Squares, if times are measured from the first observation, we have to make

$$(y_0 - a)^2 + (y_1 - a - ib)^2 + (y_2 - a - 2ib)^2 + \ldots + (y_n - a - nib)^2$$

a minimum by the variation of a and b. Differentiating with respect to a and b, we have

$$(y_0 - a) + (y_1 - a - ib) + (y_2 - a - 2ib) + \ldots + (y_n - a - nib) = 0,$$

$$(y_1 - a - ib) + 2(y_2 - a - 2ib) + \ldots + n(y_n - a - nib) = 0.$$

These give

$$(n+1)a + \tfrac{1}{2}n(n+1)ib = y_0 + y_1 + \ldots + y_n, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\tfrac{1}{2}n(n+1)a + \tfrac{1}{6}n(n+1)(2n+1)ib = y_1 + 2y_2 + \ldots + ny_n. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Eliminating a, we have

$$\tfrac{1}{6}n(n+1)(n+2)ib = ny_n + (n-2)y_{n-1} + (n-4)y_{n-2} + \ldots - (n-2)y_1 - ny_0.$$

This formula gives b, the required probable rate of increase. We may write it in the form:

$$b = \frac{n(y_n - y_0) + (n-2)(y_{n-1} - y_1) + (n-4)(y_{n-2} - y_2) + \ldots}{\tfrac{1}{6}n(n+1)(n+2)i}, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

and it will be seen that, unlike the previous formula, it takes account of intermediate as well as extreme values. But it still remains to find a simple interpretation showing why and wherefore the formula takes this particular form. This interpretation may be given in the following rule:

Take all the different pairs of observed values that can be found by combinations of the $n+1$ observations taken two at a time. Add together the total increases and divide by the sum of the time intervals which these increases represent.

We may verify this rule by exhibiting the differences and the time intervals in the following tabular form:

Differences.	*Intervals.*
$y_n - y_{n-1}$	i
$y_n - y_{n-2}$	$2i$
$y_n - y_{n-3}$	$3i$
.....	.....
.....	.....
$y_n - y_0$	ni $\quad \tfrac{1}{2}n(n+1)i$
$y_{n-1} - y_{n-2}$	i
$y_{n-1} - y_{n-3}$	$2i$
.....	.....
.....	.....
$y_{n-1} - y_0$	$(n-1)i$ $\quad \tfrac{1}{2}(n-1)ni$
.....	.....
.....	.....
$y_1 - y_0$	i $\quad i$
$ny_n + (n-2)y_{n-1} + \quad -(n-2)y_1 - ny_0$	$\tfrac{1}{6}n(n+1)(n+2)i$

We may also state the rule as follows: The total increases of value determined from each of the $\frac{1}{2}n(n+1)$ pairs of observed values taken two at a time are of equal weight.

If, instead of taking all possible pairs of observations, we take only the pairs formed by the first and last, the second and last but one, and so on, the total increases in the corresponding intervals are $y_n - y_0$, $y_{n-1} - y_1$ and so on, and the lengths of the intervals are ni, $(n-2)i$, ... With this arrangement the weights of the several intervals are proportional to the lengths of these intervals, as is evident from equation (3).

Equation (1) gives

$$a + \tfrac{1}{2}nib = \frac{y_0 + y_1 + y_2 + \ldots + y_n}{n+1}$$

or the most probable value of the quantity y at the middle of the period of observation is the arithmetic mean of the observed values, as it evidently should be.

To take a numerical example let us calculate the average annual rate of increase of the quantity of pig iron produced by Great Britain during the period 1891-1901, the annual quantities, in million tons, being as follows (from Whitaker's *Almanack*):

7·4, 6·7, 7·4, 7·3, 7·6, 8·6, 8·7, 8·5, 9·4, 8·8, 7·8.

The process may be written thus:

	Increase.	No. of years.		Sums.	Corresponding No. of years.
$d_1 = y_{10} - y_0$	0·4	10	d_1	= 0·4	10
$d_2 = y_9 - y_1$	2·1	8	$d_1 + d_2$	= 2·5	18
$d_3 = y_8 - y_2$	2·0	6	$d_1 + d_2 + d_3$	= 4·5	24
$d_4 = y_7 - y_3$	1·2	4	$d_1 + d_2 + d_3 + d_4$	= 5·7	28
$d_5 = y_6 - y_4$	1·1	2	$d_1 + d_2 + d_3 + d_4 + d_5$	= 6·8	30
			$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5$	= 19·9	110

The mean rate of increase given by the formula is $19{\cdot}9 \div 110 = {\cdot}181$ million tons per annum; whereas if the increase were calculated by dividing the difference between the first and last values by the number of years, the mean rate of increase would be only ·04 million tons. This is a very good example of the use of the method. There is very little difference between the first and last values of the observed quantity, but the remaining values show that the production was considerably greater as a rule towards the end than towards the beginning of the decade. Here, then, we have a case in which the difference between the initial and final values fails to afford an efficient measure of the general increase.

The tabular working of the above example also shows a useful method of calculating such quantities as $(1a + 2b + 3c + 4d \ldots) \div (1 + 2 + 3 + 4 \ldots)$. The numbers in the right-hand half are the sums of the corresponding numbers in the left-hand half, added up thus: "0·4 and 2·1 are 2·5 and 2·0 are 4·5 and 1·2 are 5·7 and 1·1 are 6·8." A similar method of adding up may frequently be found of use in obtaining average of marks of candidates in examinations, when the distribution of the candidates over the various percentages is known.

GEORGE H. BRYAN.

UNIVERSITY COLLEGE OF NORTH WALES,
BANGOR, WALES, 11*th October*, 1904.

THE DEVELOPMENT OF GEOMETRICAL METHODS.*

In order to give an adequate account of the progress of Geometry in the century that has just come to a close, it is important to cast a rapid glance over the condition of Mathematical Science at the beginning of the nineteenth century.

We know that during the latter portion of his life, Lagrange, somewhat weary of the researches in Analysis and Mechanics, which, however, assured for him immortal glory, took up the study of Chemistry (which he described as almost as easy as Algebra), of Physics, and also devoted himself to philosophical speculation. We almost always see a similar state of mind at certain periods in the lives of the greatest men of science. From the fresh ideas of their fertile youth, ideas which they utilise to the full in the field of their labours, they have derived all that they have any right to expect; their task is done; their mental activity is impelled towards entirely new subjects. And this impulse, be it understood, had a quite peculiar force in the days of Lagrange. The field of research opened to geometers by the discovery of the Infinitesimal Calculus appeared to be almost exhausted. It really seemed that when a few more or less complicated differential equations were integrated, and a few more chapters were added to the Integral Calculus, the very limits of Science would be attained. Laplace had completed his explanation of the system of the world, and was laying the foundations of Molecular Physics. New paths were opening for the experimental sciences, and all was ready for the astonishing development they were to receive during the nineteenth century. Ampère, Poisson, Fourier, and Cauchy himself, the creator of the theory of imaginaries, were entirely taken up with the application of analytical methods to Mechanics and Molecular Physics; they seemed to believe that apart from the new domain of which they were making so rapid a survey, the limits of Theory and Science were definitely fixed.

From the end of the eighteenth century, the subject for which we must claim the name of modern Geometry had been in a large measure contributing to the renaissance of Mathematical Science in all its branches. It had offered to research a new and fertile path, and, in particular, it had shewn in the most striking and successful manner, that general methods are not

* An address by M. Gaston Darboux, of the *Institut de France*, Sept. 24, 1904, given at the Congress of Science and Art, St. Louis, U.S.A. The author has very kindly given permission for the publication of this interesting address in the *Gazette*.

This distinguished mathematician is Permanent Secretary to the Académie des Sciences; Honorary Dean of the Faculté des Sciences. In addition to numerous Memoirs he has published: *Leçons sur la Théorie des Surfaces et les Applications géométriques du Calcul Infinitésimal*. 4 vols. in 8°. 1887-1889-1894-1896 (Gauthier-Villars). *Coordonnées curvilignes*. 1 vol. in 8°. 1898 (Gauthier-Villars).

everything in Science, and that even in the simplest subject there is much that may be done by the ingenious and inventive mind. The beautiful geometrical demonstrations of Huyghens, Newton, and Clairaut were forgotten or neglected. The fine ideas introduced by Desargues and Pascal had never been developed and seemed to have fallen on barren soil. Carnot, by his *Essai sur les Transversales* and his *Géométrie de Position*, and Monge, more especially by his beautiful theories on the generation of surfaces, had pieced together a chain the links of which had seemed to be severed. Thanks to them, the conceptions of Descartes and Fermat, the inventors of Analytical Geometry, assumed once more, in the Infinitesimal Calculus of Leibniz and Newton, the position they had lost and which they should never have ceased to occupy. As Lagrange said of Monge, "ce diable d'homme" will make himself immortal by his Geometry. And in fact, not only did Descriptive Geometry play its part in coordinating and perfecting the processes employed in all the arts, "in which precision of form is a condition of success as well as of excellence for labour and its products," but it made its appearance as the graphical translation of a general and purely rational Geometry, the fertility of which has been shewn by researches both numerous and important. At the side of his *Géométrie Descriptive* we must not forget to place his other master-piece, the *Application de l'Analyse à la Géométrie*. Nor should we forget that to Monge we owe the idea of lines of curvature, and the elegant integration of the differential equations of those lines in the case of the ellipsoid, which it was said was a matter of envy to Lagrange. We must lay stress on this character of the whole of the work of Monge. The man who reconstructed what we call modern Geometry has shewn us from the outset, and this has perhaps been forgotten by his successors, that the alliance between Analysis and Geometry was useful and fruitful, and that perhaps this alliance was a condition of success for both these branches of Mathematics.

II.

To the school of Monge we owe many geometers: Hachette, Brianchon, Chappuis, Binet, Lancret, Dupin, Malus, Gaultier de Tours, Poncelet, Chasles, etc. Among these, Poncelet ranks first and foremost. He passed over everything in the works of Monge connected with Cartesian Analysis or with Infinitesimal Geometry; he devoted his attention exclusively to the development of the germs contained in the purely geometrical researches of his illustrious predecessor. He was captured by the Russians at the passage of the Dnieper, and interned at Saratoff. During the period of enforced leisure which followed he turned his attention to the proof of the principles which he developed in

his *Traité des Propriétés projectives des Figures*, published in 1822, and in the great memoirs on polar reciprocals and harmonic means which date back to about this time. It may therefore be said that modern Geometry was born at Saratoff. Returning to the ideas which had been neglected since the days of Pascal and Desargues, Poncelet simultaneously introduced homology and polar reciprocals, thus at the outset throwing into relief the ideas which Science had been maturing for 50 years.

Poncelet's methods were not enthusiastically received by French analysts, being in a sense opposed to the processes of Analytical Geometry. But their importance was so great and their novelty so striking that it was not long before they were thoroughly examined. Poncelet alone had discovered the principles; on the other hand, several geometers almost simultaneously began to study them from every point of view and succeeded in deducing from them the essential results which they implicitly contained.

At this time, Gergonne was the editor of a periodical which is of inestimable value to us in the history of Geometry. The *Annales de Mathématiques*, published at Nîmes from 1810 to 1831, had been for more than fifteen years the only journal in the world exclusively devoted to mathematical research. Gergonne was in many respects a model editor for a scientific journal; but he had the defects of his qualities. He collaborated, often against their will, with the authors of the memoirs which were sent to him; he altered their papers, and sometimes made them say more or less than they had intended. However that may be, he was greatly struck with the originality and the scope of Poncelet's discoveries. A few simple ways of transforming figures were already known in Geometry; homology had even been used in the plane; no one but Poncelet had extended it to space, and no idea seemed to exist of the wealth and fertility of the method. Besides, all these transformations were *ponctuelles*, point was made to correspond to point. By the introduction of polar reciprocals, Poncelet became an inventor of the highest rank; he gave the first example of a transformation in which to a point corresponded something else than a point. Every method of transformation enables us to multiply our theorems, but the method of polar reciprocals had the advantage of bringing into correspondence two entirely different propositions. This was essentially something new. To throw this into relief Gergonne invented the plan, which has since been adopted with so much success, of writing correlative propositions in two columns. His also was the idea of substituting for Poncelet's demonstrations, which required the assistance of a curve or surface of the second order, the famous

principle of duality, the significance of which, though somewhat vague at first, was sufficiently cleared up by the discussions which took place on the subject between Gergonne, Poncelet, and Plücker.

Bobillier, Chasles, Steiner, Lamé, Sturm, and many others, whose names escape me, assiduously collaborated with Poncelet and Plücker on the *Annales*. When Gergonne was appointed Rector of the Académie de Montpellier in 1831, his publication of the journal was interrupted. But his success in creating a new form of mathematical literature, and the taste for research which he had done so much to develop, had already begun to bear fruit. Quetelet had just started the Belgian *Correspondance mathématique et physique*. Crelle in 1826 had published the first sheets of the famous journal which bears his name, and in which are to be found memoirs by Abel, Jacobi, and Steiner. A considerable number of formal treatises also appeared at this time, in which the principles of modern geometry were being expounded and developed in a masterly manner.

First, there was in 1827 the *Barycentrisches Calcul* of Möbius, a really original work, remarkable for the depth of its conceptions and the lucidity and rigour of its exposition. In 1828 appeared Plücker's *Analytisch-geometrische Entwickelungen*, the second part of which was published in 1831. These were very soon followed by the same author's *System der analytischen Geometrie* (Berlin, 1835). In 1832 Steiner published his *Systematische Entwickelung der Abhängigkeit der geometrischen Gestalten von einander*, and in the next year appeared his *Geometrische Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises*, in which was proved by the most elegant examples a proposition of Poncelet's relative to the use of a single circle in geometrical constructions. Finally, in 1830, Chasles sent to the Académie de Bruxelles, which by a happy inspiration had offered a prize for the best essay on the principles of modern geometry, his celebrated *Aperçu historique sur l'origine et le développement des méthodes en Géométrie*. This was followed by the *Mémoire sur deux principes généraux de la Science: la dualité et l'homographie*, which was not published until 1837.

Time would fail me were I to endeavour to offer an adequate appreciation of these noble works, or on such an occasion as this to give an account of each of them. Such an attempt would lead us to a new verification of the general laws of the development of science. When the moment is ripe, when the fundamental principles are recognised and enunciated, nothing can check the march of ideas; the same or nearly the same discoveries are made almost simultaneously, and in all parts of the world. But although I shall not venture on a discussion of

this kind, which would be of little use, and might give but little pleasure, it is important that I should try to throw some light upon a fundamental distinction in the tendencies of the great geometers who, about 1830, gave to Geometry an impulse hitherto unknown.

III.

Some, like Chasles and Steiner, who devoted the whole of their lives to researches in Pure Geometry, drew a distinction between what they called *synthesis* and *analysis*; and, adopting in their general scope, if not in their detail, the tendencies of Poncelet, endeavoured to found an independent theory, a rival to the Cartesian Analysis.

Poncelet did not rest content with the inadequate resources furnished by the method of projections. He therefore imagined the famous *principle of continuity*, which gave rise to such long discussions between him and Cauchy. The principle is an excellent one when properly enunciated, and may be of the greatest service. Poncelet made the mistake of refusing to present it as a simple consequence of Analysis; Cauchy, on the other hand, would not recognise that his own objections, valid no doubt in the case of certain transcendental figures, lost their force in the applications made by the author of the *Traité des propriétés projectives*. Whatever opinion may be held on the subject of such a discussion, it at least shewed, in the clearest manner, that Poncelet's geometrical system rested on an analytical basis, and we know besides, from the unfortunate publication of the Saratoff manuscripts, that the principles which served as a foundation for the *Traité* were established by the aid of the Cartesian Analysis. Poncelet had abandoned Geometry for Mechanics, in which his work had a preponderating influence. Chasles, his junior, for whom had been created in 1847 a chair of Higher Geometry in the *Faculté des Sciences de Paris*, set to work to build up a geometrical theory entirely independent and autonomous. This he expounded in two works of the highest importance, his *Traité de Géométrie Supérieure* (1852), and the *Traité des Sections Coniques*, unfortunately left unfinished, the first part alone appearing in 1865.

In the preface to the former he very clearly indicates the three fundamental points which enable the new theory to participate in the advantages of analysis, and seem to him to mark some progress in the cultivation of the science. They are:

(1) The introduction of the principle of signs, which simplifies both enunciations and proofs, and gives to the analysis of Carnot's transversals all the scope of which they are susceptible.

(2) The introduction of imaginaries, which supplements the

principle of continuity, and furnishes demonstrations as general as those of analytical geometry.

(3) The simultaneous demonstration of correlative propositions, *i.e.*, propositions which correspond in virtue of the principle of duality.

Chasles thoroughly discusses homography and correlation in his work, but in his exposition he systematically avoids the use of the transformation of figures. These, he thinks, cannot supplement direct proofs, because they mask the origin and real nature of the properties obtained by their means. There is something in this criticism, but the march of science shews us that it errs on the side of severity. Even if it often happens that transformations, when employed without discrimination, uselessly multiply the number of theorems, yet we must not forget that they often assist us to a better realisation of the propositions to which they are applied. Did not the use of Poncelet's projection, which led to the fruitful distinction between projective and metrical properties, make us recognise the great importance of this anharmonic ratio, the essential property of which is to be found in Pappus, and the fundamental rôle of which did not appear in modern geometry until fifteen centuries later?

The introduction of the principle of signs was not the novelty Chasles supposed when he wrote his *Traité de Géométrie Supérieure*. Möbius in his *Barycentrisches Calcul* had already followed up a *desideratum* of Carnot's, and had used signs in a very wide and precise manner, defining for the first time the sign of a segment and even of an area. Later on he succeeded in extending the use of signs to lengths which are not measured off on the same straight line, and to angles which are not formed around the same point. In addition, Grassmann, whose mind presents so many points of analogy with that of Möbius, was necessarily compelled to use the principle of signs in the definitions which serve as a basis for his remarkably original method of discussing the properties of extension.

The second characteristic assigned to his system of geometry by Chasles is the use of imaginaries. Here his method was really new, and he was able to illustrate it by examples of the greatest interest. Admiration will always be awarded to the beautiful theories he has left us on the confocal surfaces of the second degree, in which every known property and many new properties, as varied as they are elegant, are derived from the general principle that these surfaces are inscribed in the same developable which is circumscribed to the circle at infinity. But Chasles only introduced imaginaries by their symmetrical functions, and was therefore unable to define the anharmonic ratio of four elements when they cease to be real, in whole or in

part. If only he could have established the idea of the anharmonic ratio of imaginary elements, a formula which is to be found in the *Géométrie Supérieure* (p. 118, new edition) would have immediately enabled him to give the beautiful definition of an angle as the logarithm of an anharmonic ratio—a definition which enabled Laguerre, my lamented colleague, to find the long-sought, complete solution of the problem of the transformation of those relations which contain both angles and homographic and correlated segments.

IV.

While Chasles, Steiner, and later, von Staudt, were endeavouring to build up a theory that would be a rival to Analysis, and were, as it were, erecting altar against altar, Gergonne, Bobillier, Sturm, and Plücker in particular, were perfecting the Cartesian Geometry, and were constructing an analytical system in some measure adequate to the discoveries of the geometers. To Bobillier and Plücker we owe what is known as the method of *Abridged Notation*. In the last volumes of Gergonne's *Annales* are to be found a few really original pages from the pen of the former. Plücker had begun to develop the method in his first volume, very soon to be followed by a series of works in which the foundations of modern geometry are deliberately established. To the same investigator we owe tangential and trilinear coordinates, used in homogeneous equations, and also the canonical forms, the validity of which is recognised in the fruitful but sometimes deceptive method known as the *enumeration of constants*. All these timely discoveries were to infuse new blood into the Cartesian Analysis; they enabled it to give their full significance to those conceptions which the so-called *synthetic* geometry had not completely grasped. Plücker—and with his name it is only fair to couple that of Bobillier, carried off by an untimely death—must be regarded as the first to familiarise us with those methods of modern Analysis, in which the use of homogeneous coordinates enables us to treat simultaneously, without the reader's knowledge, as it were, a figure, and all the figures that are deduced from it by homography and correlation.

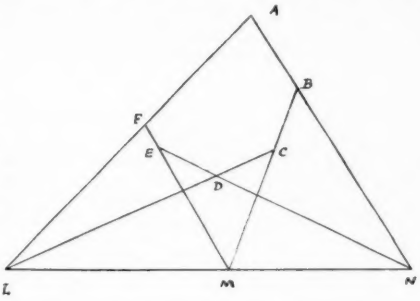
(To be continued.)

MATHEMATICAL NOTES.

156. [L^2. 7; 14. a.] *To prove by Pascal's theorem that the straight lines meeting three non-intersecting straight lines generate a conicoid, i.e. a surface every plane section of which is a conic.*

Let P_1, P_2, P_3; p_1, p_2, p_3 be two sets of non-intersecting straight lines in space, each line of one set meeting all of the other set. Let them meet

any plane, not containing one of them, in the points A, C, E, B, D, F. Let AF, CD; FE, BC; ED, AB meet respectively in L, M, N. Denote the intersections P_1p_2, P_2p_3, P_3p_1 by X, Y, Z. Now X, Y, L lie in one straight line for they will all be found in both the planes P_1p_3 and P_2p_2, and therefore lie on their intersection.



Similarly Y, Z, M lie in the planes P_3p_3 and P_2p_1, and X, Z, N lie in the planes P_3p_2 and P_1p_1. Therefore L, M, N are the points in which the sides of the triangle XYZ meet the plane of $ABCDEF$, and are therefore in one straight line, so that by Pascal's theorem A, B, C, D, E, F lie in a conic. Since five points determine the conic, we may consider F, and therefore p_3, as variable, and any straight line meeting P_1, P_2, P_3 traces out the conic $ABCDE$.

W. H. BLYTHE.

157. [P. 3. b. a.] *To find the relation between two maps of the same contour on the gnomonic projection.*

Let l be any line touching at A a sphere whose centre is O, and let m be the parallel line through O. Let ξ be the projection from O on the sphere of any curve α on the tangent plane p at A. Let ζ be the reflection of ξ in a plane through m making an angle θ with the plane mA and cutting p in the line n. Let η be the reflection of ζ in the plane mA. Then η is obtained from ξ by rotating ξ about m through an angle 2θ. Let β, γ be the projections of η, ζ on p. Then α, β are two different gnomonic projections of the same spherical curve. Draw VAW perpendicular to l in the plane p cutting n in W, such that VOW is a right angle. Then the line joining corresponding points of α and γ evidently passes through V and is divided harmonically by V and its intersection with n. Hence β is the reflexion in l of the harmonic homologue of α, V being the centre, and n the axis of homology.

Since the theorems given in the *Math. Gazette*, Jan. 1904, p. 383, and March, 1904, p. 7, on successive inversion with respect to coaxial circles can be proved by aid of the "relation between two maps of the same contour on the stereographic projection" (*Math. Gazette*, May 1904, p. 33; Oct. 1904, p. 88), we see that in the theorems referred to we may replace "inversion with respect to one of a system of coaxial circles" by "the operation of taking the harmonic homologue of a figure," "the axis of homology being one of a system of parallel lines and the centre its pole with respect to a fixed circle." An independent proof may be readily given.

HAROLD HILTON.

158. [P. 3. b.] *On a Theorem in Inversion.*

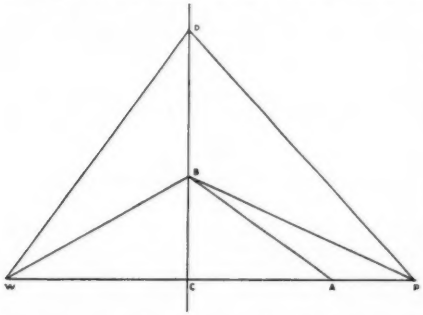
Since two points P, P' on a sphere which are reflexions of each other in the plane of a great circle s are projected stereographically into two points inverse with respect to the projection of s, the theorem 153 (p. 88) becomes: the stereographic projections of two figures on the sphere which are reflexions

of each other in the plane of s from P and P' respectively are enantiomorphous —which is obvious (cf. 154, p. 88). If we use the gnomonic projection instead of the stereographic, we have: If $\alpha(F)$ denote the harmonic homologue of any figure F, A being the centre of homology and the polar of A, with respect to a given circle the axis of homology, then $\pi(F)$ and $\pi'(F')$ are enantiomorphous, where $\alpha(F)=F$, $\beta(A)$ is at infinity, $\beta(P)=P'$, and AB passes through the centre of the given circle (cf. Note 157).

HAROLD HILTON.

159. [P. 3. b. a.] *A note on the gnomonic projection.*

Let $B, C, D, P, \ldots$ be the projections from O on the tangent plane at A of the points $B', C', D', P', \ldots$ on a sphere whose centre is O. (B may be conveniently marked with a × or a ○ according as the angle BOA is acute or obtuse.) Draw $PACW$ perpendicular to BD meeting BD in C; let $WC^2=AC^2+AO^2$ and $CA \cdot AP=AO^2$. Then the angle $BWD=B'OD'=$ the angle between the great circles $B'P'$, $D'P'$, since $B'P'$, $D'P'$ are quadrants, and any point on BD is equidistant from O and W. These facts enable us to measure the angle and sides of any spherical triangle whose gnomonic projection is given. They also enable us to solve graphically any spherical triangle; for the gnomonic projection of any spherical triangle on the tangent plane at a vertex can be readily found when the three sides, or the two sides meeting at that vertex and any angle are given.



Moreover the projection enables us to prove the usual formulae of spherical trigonometry. For example: since

$$\sin BAC=\frac{BC}{BW}\div\frac{BA}{BW};\ \cos BAC=\frac{AC}{AO}\div\frac{AB}{AO};\ \tan BAC=\frac{CB}{CW}\div\frac{CA}{CW};$$

we have the following formulae for a spherical triangle right-angled at C: $\sin A=\sin a \,.\, \operatorname{cosec} c$; $\cos A=\tan b \,.\, \cot c$; $\tan A=\tan a \,.\, \operatorname{cosec} b$.

HAROLD HILTON.

160. [L¹. 3. a.] *On the Equation to the axes of the general conic.*

The points on the ellipse $x^2/a^2+y^2/b^2=1$ the normals at which cointersect in a given point (ξ, η) are determined as the intersections of the ellipse with a definite rectangular hyperbola $a^2\xi/x-b^2\eta/y=a^2-b^2$. When $\xi=0$, $\eta=0$ this R.H. degenerates into the two axes of the ellipse.

Similarly the points on the general conic the normals at which cointersect in a given point (ξ, η) lie on the R.H.

$$(ax+hy+g)/(x-\xi)=(hx+by+f)/(y-\eta).$$

Hence making (ξ, η) the centre of the curve, the equation to the axes of the conic is

$$(ax+hy+g)(y-F/C)=(hx+by+f)(x-G/C).$$

R. F. DAVIS.

161. [R. 4. d.] *Graphic Construction for the Central Axis of two Non-intersecting Forces.*

Let P, Q be the forces, α the angle between their directions, d the shortest distance between them.

Draw AC, CB representing P, Q in magnitude and direction, so that $ACB = 180° - \alpha$.

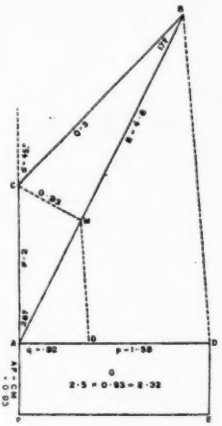
Then AB represents in magnitude and direction the resultant force R. Draw CM perpendicular to AB.

In any direction measure off AD equal to the shortest distance d. Divide AD in the ratio of AM to MB, and let O be the point of section.

Then AO, OD are the segments into which the shortest distance is divided by the central axis, OA being the distance from Q (which we shall call q) and OD the distance (p) from BP.

The moment of the resultant couple is represented by the area of a rectangle $ADEF$ whose base AD is the shortest distance d, and whose height AF is equal to CM.

The construction in the figure was made for $P=2$, $Q=3$, $\alpha=45°$, $d=2·5$, the lengths in the original figure being measured in inches.



On measurement the following rough results were found:

$$AB=4·8 \qquad \angle CAB=28° \qquad \angle CBA=17°,$$
$$AO=0·92 \qquad OD=1·58 \qquad CM=0·93,$$

leading to the following conclusions:

Resultant force $=4·8$ units, making angles 28° with the force of 2 units and 17° with the force of 3 units.

Central axis is 1·58 inches from the former and 0·92 inches from the latter force, measured along the shortest distance.

Moment of resultant couple $=2·5\times0·93=2·32$. G. H. Bryan.

REVIEWS.

Etudes sur les quantités mathématiques; grandeurs dirigées—quaternions. By Claro Cornelio Dassen. Pp. vi, 136. 5 francs. 1904. (A. Hermann, Paris.)

This volume consists of two parts, the concept of quantity, pp. 1-34, and directed quantities, pp. 35-112, followed by appendices on analysis based uniquely and entirely on the notion of the whole number, on angles and arcs, "dits imaginaires," and on space of four dimensions and quaternions.

Part I. contains an interesting account of the history of the fundamental symbols of arithmetic and of various systems of scales of notation. The associative, distributive and commutative laws of arithmetical operations are examined in considerable detail. Chap. I., Part II., is occupied with a discussion of the laws of combination of "quantités dirigées sur un droit ou en deux sens," and here as elsewhere throughout the work there are numerous references to what Clerk Maxwell called the "archaeology" of the subject. The second chapter deals with directed quantities in a plane, and contains critical examinations of the various interpretations that have been assigned to the square root of minus one. Directed quantities in space are considered in Chap. III. (pp. 86-112), which is chiefly occupied with quaternions, though a page or two are devoted to the Ausdehnungslehre, and the Icosian calculus is mentioned. In introducing the conception of the multiplication of directed quantities in space, the author says, "Si l'en veut donc étendre la théorie de la multiplication aux vecteurs dirigés dans l'espace, on est obligé de renoncer à la conservation de certaines propriétés

combinatoires de cette opération; mais comme—il ne faut pas l'oublier—les définitions et les suppositions que l'on fera doivent être subordonnées à leur utilité pratique, il faut se garder de laisser à la fantaisie le choix des hypothèses; au contraire, on tâchera de conserver les propriétés les plus importantes, celles qui se prêtent aux combinaisons des calculs; ces propriétés sont sans contredit l'uniformité, l'associativité et la distributivité par rapport à l'addition, car la commutativité et l'existence d'un module ne sont pas nécessaries pour le mécanisme du calcul." Would that the "fantaisie" of the writers in the *Encyklopädie der math. Wiss.* had been controlled by such common sense; the preservation of the associative law would lead to the extinction of most of the numerous family of systems of vector analysis. C. J. JOLY.

Leçons Élémentaires sur la théorie des Groupes de Transformations. Par G. VIVANTI. Traduites par A. BOULANGER. Pp. 296. (Paris, Gauthier-Villars.) 1904.

Nothing is stated in the title or preface of this book as to its previous appearance in Italian, or as to whether it has been translated under the supervision of the author. In the absence of information to the contrary it is assumed to be an authorised reproduction in French of the course delivered by Professor Vivanti at Messina in the years 1897-8, and at the time lithographed.

It must have been a valuable course of lectures, calculated to give hearers a true notion of the fundamentals and salient features of Lie's great theory, in practically unmodified order and form, and to encourage those who could to complete the study of the volumes from which it was in effect culled. As a printed treatise it will be of the same utility to a larger circle of students, and especially so in a new language. But it cannot be considered a work of an enterprising or original character. The straightforwardness of the procedure by which it has been constructed is described in the following extract from the preface: "Nous parcourons les deux premiers volumes du grand ouvrage de Lie, en cherchant à montrer les grandes lignes et à fixer les points fondamentaux de la nouvelle doctrine; nous consacrerons aussi quelque pages à l'application des vues de Lie aux Equations differentielles ordinaires." To this should have been added that in the pages last referred to the method has been the same as in the others, with Scheffers now for guiding authority upon Lie's views. It is the way of a painstaking tutor, who goes through a voluminous standard treatise with a red pencil, marks the sections suitable for a first reading, and then reduces the marked portions to a manuscript for the benefit of his pupils. He often thinks, rightly enough, that this would be useful printed, but is generally fastidious.

The first 132 pages of the book are devoted to the general theory of groups of transformations, in other words to Lie-Engel's first volume. The extracts made from this are judicious and fairly ample. Then come 61 pages on differential equations of the first order; and, lastly, 99 pages on contact transformations.

Granted that an editing of a first course taken from Lie is to be welcomed, granted also that a first course for editing has been chosen with discretion, the question remains whether the variations of expression introduced into the proofs of propositions selected do or do not tend in the direction of simplicity. Most of the variations here are merely matters of paraphrase and notation. The rest are of two kinds: (1) removals of extreme elaboration and painfully minute attention to rigour and completeness, and (2) introductions of pieces of preliminary analysis, in cases where Lie's arguments had directly used, as known or proved at an earlier stage, what could hardly be deemed part of the common knowledge of beginners or had been excluded. These are to be commended. Consequently the book, regarded as an unambitious one, is quite good of its sort.

A good preparatory work, however, may be made much better by the introduction into it of numerous elementary illustrative examples. Here there are practically none, except about a score (taken without exception from Scheffers) in the differential equations section. E. B. ELLIOTT.

Vorlesungen über Technische Mechanik. Zweite Band. **Graphische Statik.** By Dr. AUG. FOPPL, Professor at the Munich Technical Institute. Pp. xii and 472. (Teubner.) 2nd edition. 10 marks.

Professor Foppl's four volumes of Vorlesungen are devoted to elementary mechanics, graphic statics, elasticity, and dynamics respectively, and aim at

giving a student some idea of modern advanced methods, in their application to certain technical problems, without employing very elaborate mathematical apparatus.

In the present volume, after a brief sketch of the rise of graphical methods, the author explains the scope of his work. He does not confine himself to the purely graphical standpoint of Cremona, in whose hands the subject is as it were a branch of projective geometry, but uses the methods of the calculus freely when convenient. On the other hand, the author declines to discuss in the lecture-room questions as to the material, design, and mode of erection of particular structures which ought to be dealt with at the works.

One object which the author keeps in view from the first is that the student should become accustomed to deal with forces in space. Thus, for instance, instead of a bare statement of the parallelepiped of forces, a graphical solution is given of the problem of resolving a force acting at a point into components acting along three given straight lines through the point, specified by their plan and elevation.

An excellent account of Bow's method of dealing with frames, the usual link polygon constructions for finding centres of gravity and moments of inertia, and the leading propositions on the composition and resolution of forces in space follow in order.

The features, however, which present most novelty to an English reader occur in the chapter on the elastic deformation of frames. In this chapter a simple graphical method (which appears to be little known here) of investigating the elastic displacements of the joints of a frame is given, which is said to be due to M. Williot (*Seyrig*; *Statique Graphique*). The question is one of considerable importance in the case of a structure of the cantilever type. Many readers will recollect the difficulty which arose in getting the central girder of the Forth Bridge into position.

An application after Maxwell (cf. Routh's *Statics*, vol. i., pp. 161-162) of the principle of virtual work to finding the stresses in the bars of a statically indeterminate frame gives rise to some interesting examples.

Special attention is given to those frames for which, though they are not statically indeterminate, the ordinary method of drawing a Bow's diagram fails and some special artifice becomes necessary. An instance of the latter class, of which the so-called Belgian roof is the most familiar example, occurred in the Mathematical Tripos this year. Those exceptional cases in which a frame is self-strained though it contains no redundant bars are also considered. They were first noticed by Professor Crofton (*Proc. Lon. Math. Soc.*, vol. x.), who showed that a frame consisting of a Pascal's hexagon with its three diagonals might be self-strained. Professor Crofton alludes to the surprise which Pascal's hexagon must feel at finding itself involved in a discussion on applied mechanics.

Professor Foppl's work abounds in interesting applications, and no review would do justice to it which failed to call attention to these and to the superlatively excellent diagrams.

C. S. JACKSON.

Elements of Plane Surveying. By SAMUEL MARX BARTON, Ph.D., Professor of Mathematics in the University of the South. Pp. viii and 255. (Boston: D. C. Heath & Co.) 1904.

The making of a survey for an underground railway beneath the streets of a great city raises some delicate problems, of which an interesting account will be found in the *Proceedings of the Institute of Civil Engineers* (vol. clvi., p. 376), but apart from such peculiar cases a topographical survey, as distinguished from a valuation of a farm or estate, is in England nowadays, owing to the existence of the large scale ordnance maps, a comparatively rare event.

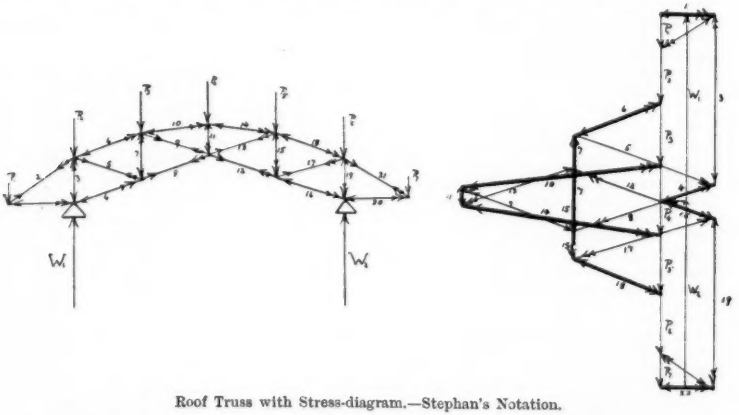
In the United States it appears, however, that the making of a survey is still an important branch of the professional duties of a surveyor, and in the present work the author has described the principal methods and instruments used in making an ordinary survey of a small tract of ground. The methods of geodetic surveying, and the photographic processes employed by Colonel Laussedat, do not come within the author's plan, but in a future edition a reference to Sir Howard Grubb's "ghost" sights and levels (*Trans. R.S., Dublin*, vii. 385) might prove useful.

An interesting substitute for the compass is described in Saegmuller's solar attachment to a theodolite, which automatically carries out the process given in books on navigation of finding compass error by observation, thus dispensing with the necessity for any computation. An auxiliary telescope is set, by means of the known latitude and the known declination of the sun, into such a position relative to the principal telescope that when the auxiliary telescope is on the sun the principal telescope, and with it the zero line on the bottom plate of the theodolite, are necessarily in the meridian. The true N. and S. line thus determined enables the observer to read off the true bearing of any object. The same instrument furnishes a simple means of finding the time by observation. A table annexed gives the effect of an error of one minute in latitude or declination on the resulting azimuth. A chapter is devoted to modern methods of "stadia" surveying, and the work concludes with a set of tables, accompanied by some notably careful and intelligible directions for their use. C. S. JACKSON.

Die Technische Mechanik. First Part: Mechanics of Rigid Bodies. By P. STEPHAN. Pp. 344. (Teubner, Leipzig.)

In the preface of this work the author states that his book is primarily intended for use in Schools of Machine Design. Statics is treated of before Dynamics, partly in order that the most simple technical questions, such as friction, may be considered as early as possible, but chiefly in order that too many new ideas and definitions may not be brought forward at the commencement.

The use of the Calculus is avoided throughout the volume, and "dodges" for circumventing its use are freely indulged in. Thus two pages (66 and 67) are devoted to the determination of the Centre of Gravity of a parabolic segment. As another illustration mention might be made of the proof of the

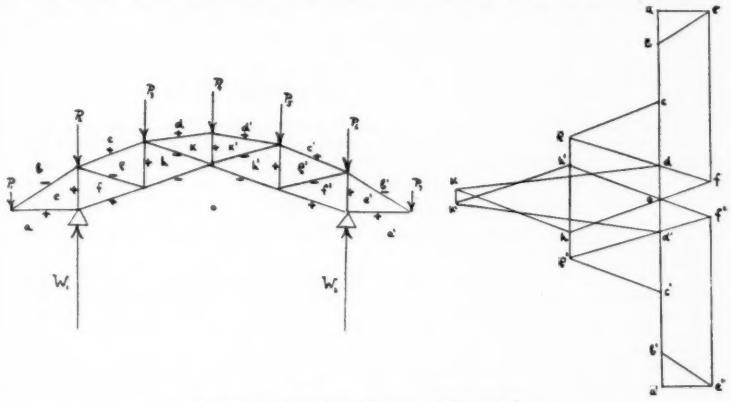


Roof Truss with Stress-diagram.—Stephan's Notation.

formula $T = T_0 e^{\mu\alpha}$ for the tension of a rope coiled round a rough surface. The proof, which we do not recollect having seen before, is ingenious and obtains the well-known result as the limiting value of $T_0\left(1+\mu\frac{\alpha}{n}\right)^n$ when n is increased indefinitely (pp. 169-171). Proofs of this kind are going more and more "out of fashion" in English text-books, and in this we are wholly in sympathy. In "dodging" the Calculus and attempting to find an easier path, we find that at each step the difficulties get greater and greater. While in mastering the Calculus all the difficulties lie in the initial stages, and when the fundamental notions are fairly grasped the subsequent path is an easy one. When all is said and done, the "dodge" is nothing more or less than the Calculus itself,

though it is dressed up so as to look like a piece of commonplace Algebra or Trigonometry.

It is a matter of the greatest surprise that Bow's notation for force-diagrams of framed structures has not been universally adopted in foreign text-books. The confusion produced by the multiplication of arrow heads in the bars of the roof truss on p. 204 seems almost hopeless. The stress-diagram as given by Stephan for this roof is here reproduced, and for comparison the same diagram with Bow's notation. Comment seems superfluous. The struts and



Roof Truss with Stress-diagram.—Bow's Notation.

ties are distinguished by positive and negative signs placed beside the bars of the frame. Stephan, on the other hand, uses thick lines in the force-diagram to denote the struts. If the force-diagram is actually to be used for the measurement of the stresses, this can only lead to faulty results. There would not be the same objection to indicating the corresponding bars of the frame-diagram by thick lines, and, in some ways, this is preferable to using the more usual positive and negative signs. This, however, might be criticised on the score that a bar might be a strut for one system of loading and a tie for another system.

About 200 examples are worked out in the text, many of them showing a pleasing originality and freshness which is sadly to seek in our ordinary school text-books. Nearly all the examples are accompanied by clear diagrams, in which the direction of the acting forces and, in many cases, also the linear dimensions and angles are also given. These diagrams are invaluable: they make mistakes arising from ambiguous or cumbrous expression in the wording of the question almost impossible.

On page 221 is given a graphical Time-table for the trains that run between Furstenwalde and Guben by way of Frankfort (on the Oder), a distance of about 80 kilometres. It is well known that these graphical time-tables are largely employed in railway offices for interpolating special trains and for many other purposes. A great deal of information and instruction may be obtained from the inspection of such a time-table.

The book is one which it is a delight to handle and read, as the print is large and the diagrams excellent. We look forward to the second part, which is to treat of the Mechanics of Liquids, with considerable interest.

R. M. MILNE.

Über eine zeitgemässe Umgestaltung des Mathematischen Unterrichts an den höheren Schulen. By F. KLEIN. Pp. 82. 1904. (Teubner.)

This is a set of five addresses delivered at the Easter vacation course for mathematical and physical schoolmasters at Göttingen in 1904, supplemented by

reprints of four earlier papers. The author pleads hard for the early introduction of the notation of the calculus into the school curriculum. The principles of the calculus have long ago been admitted into the school course, and it is illogical (to say the least of it) to obtain the critical values of a function by "Schellbach's rule" and to discriminate between maxima and minima by "Fermat's rule," when the methods are those of the calculus pure and simple. If a mathematical pupil becomes gradually familiar with the calculus at an early stage of his career, he will not have to experience a "revolution of ideas" later on. Pupils who are to study law, medicine, chemistry, etc., instead of mathematics at the University need the proposed reform still more urgently. They have no time to continue their mathematical studies at the University, and so under the present system are for ever excluded from the more important mathematical developments of their subject. Their only hope is to read some abridged treatise of the *Calculus for Chemists* type, which they can never really understand, since they have had no suitable training at school.

Other reforms are suggested; artificial problems should be excluded, the graphical illustration of simple functions should be taught early, more stress should be laid on the applications of mathematics (especially to physics), and so on. The practical difficulties involved in the change are given a fair discussion, but this part of the book will naturally be of less interest to English readers. H. H.

Plane Trigonometry. By JAMES M. TAYLOR. Pp. 171. 3s. 6d. 1904. (Boston: The Athenæum Press.)

The chief merit of this book is the large number of examples of the "heights and distances" type. These are first introduced at the very beginning of the book, and are well calculated to stimulate a pupil's interest in the subject. Circular measure is postponed till after the solution of oblique triangles, and is followed by a discussion of graphs, limits, complex numbers, etc., which is too brief to be of any value. "Csc A," "Center," "quality of a line" have a rather curious appearance to English eyes. H. H.

Theoretical Geometry for Beginners. Part IV. By C. H. ALLCOCK. Pp. 224. 1s. 6d. 1904. (Macmillan & Co.)

The first half of this book covers Euclid VI., and is well illustrated by an abundant supply of examples of varying difficulty. The latter half supplies a very interesting introduction to the theory of harmonic ranges, poles and polars with respect to a circle, centres of similitude, inversion, maxima and minima treated geometrically, and envelopes. The theory of ranges other than harmonic would be a welcome addition to future editions; the book is sufficiently attractive to make the reader wish for more. H. H.

Elementary Trigonometry. By C. H. P. MAYO. Pp. 204. 1904. (Longmans, Green & Co.)

This book is divided into a practical and a theoretical section. The former begins with a definition of sine, cosine, and tangent applicable to angles of any magnitude and a table of these functions for acute angles. This table is at once applied to the finding of heights and distances and to the solution of oblique triangles by splitting them up into two right-angled triangles. The second section begins with the definition of cosecant, secant, and cotangent, and ends with circular measure and the obtaining of the trigonometrical functions of 30°, 45°, etc. It is a pity that so little stress is laid on the more usual methods of solving triangles. H. H.

Elementary Plane Geometry. By V. M. TURNBULL. Pp. 136. 2s. (Blackie & Son.)

This excellent text-book is based on the report of the Cambridge Syndicate (1903), and is intended for students who have already been through a preliminary course of experimental work. The proofs given are clear and concise, but rigorous: *e.g.* Euclid I. 5 is proved by folding about the bisector of the vertical angle, but it is first shown that this bisector exists. The whole book is illustrated by a plentiful and well-chosen supply of practical and theoretical exercises. The proofs of the theorems alone are given, the proofs of the constructions being left

to the pupil. The ground covered is that of Euclid I. to VI.; the theorems of Euclid II. are rightly considered as illustrating algebraic formulæ, not as proved by those formulæ. H. H.

Elementary Pure Geometry, with Mensuration. By E. BUDDEN, M.A., B.Sc. (Chambers.) 1904.

Mr. Budden's book is full of interest, but the method of treatment is probably too condensed for due assimilation by a beginner. Moreover, in some cases the proofs are incomplete, thus in obtaining the locus of points (in a plane) equidistant from two given points he proves that all points on the perpendicular bisector satisfy the condition, but does not show that points not on that line fail to satisfy the condition. This same neglect of the second of the two conditions for a complete locus occurs elsewhere, and spoils the logic of proof based on such loci.

Nevertheless, for older students who wish to have all their geometrical mathematics comprised in a single book, and who are able to correct such slips on their own account, the book is likely to be very useful. It is most comprehensive, including the substance of Euclidean Plane and Solid Geometry with extensions to centres of similitude, antiparallels, etc., elementary trigonometry to solution of triangles, harmonic and anharmonic ratios, and other concepts of modern geometry, geometry of conic sections, and mensuration of solids.

The printing is clear, and the collection of examples excellent, and although some of the earlier proofs may be found fault with on the score of rigorousness, the terse vigour of the proofs in general is most remarkable, and in this respect the book will be found invaluable. Mr. Budden's treatment of "ratio" is worthy of minute consideration, as it appears to cover the case of incommensurables with easy naturalness. The book should be in every mathematical library.

A. LODGE.

Mathematical Problem Papers. By E. M. RADFORD. Pp. 203. 1904. (Cambridge University Press.)

This is a collection of one hundred problem papers, each consisting of twelve questions, intended for the use of candidates for entrance scholarships at Cambridge. A large number of the problems are original and the rest have been compiled from various University and College examination papers "other than entrance scholarship papers," so that the work does not enter into competition with the well-known volumes of scholarship papers issued by the University Press.

Indeed what must strike anyone looking through this collection is the unfamiliarity of the great bulk of the questions. Occasionally one comes across a question from the *Educational Times* (lxxxvi., 3), or the *Gazette* (xlix., 7), or Roberts' *Dynamics* (viii., 11), or Besant's *Dynamics* (xlii., 11), or an old Tripos question (xlix., 11). For the most part, however, the old, tried friends of one's youth have quietly dropped out, and their places are filled by strangers of a later generation and of the most uncompromising aspect. Though, of course, there are scattered here and there a few easy questions, the papers, which are graduated in degree of difficulty, proceed from ordinary flagellation to chastisement with scorpions. One might safely predict that any scholarship candidate who had been nurtured upon, and could assimilate, such strong diet as is here provided, would not only be absolutely certain of success, but could also count upon a respectable place in the Tripos. It might be possible to work steadily through the first half of the book (one paper a week) in the year before going up to Cambridge, and to finish the other half during the first year of residence. This, of itself, would almost prove a sufficient problem equipment for the first four days of the Tripos.

The geometrical questions are sometimes peculiar statements of old properties. For instance (xl., 8) "Prove that all parabolas with a given self-conjugate triangle touch four fixed straight lines." The author has evidently followed recent work in the *Gazette* when he gives (xciii., 1) "Prove that the operations of inversion with respect to two circles are commutative only when the circles c ut orthogonally." One of H. E. Dudeney's puzzles appears in (xlix., 1) "Given two points in a plane find a compass construction for the mid-point of their join." This little study in stigmatic geometry will well repay attention.

The analytical portions of the work are most distinguished. In (xxiii., 7) the equation is given to the lines joining the origin to the four points of intersection of two conics $u_0+u_1+u_2=0$ and $v_0+v_1+v_2=0$ in the simple form.

$$\begin{vmatrix} u_0 & u_1 \\ v_0 & v_1 \end{vmatrix} \cdot \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_0 & u_2 \\ v_0 & v_2 \end{vmatrix}^2$$

In (xliii., 8) the equation to the four normals from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is stated as

$$\{a^2(x-h)^2+b^2y-k)^2\}(kx-hy)^2=(a^2-b^2)^2(x-h)^2(y-k)^2,$$

which is easily obtained by identifying the equations $y-k=\lambda(x-h)$ and $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$.

A great "tour de force" is displayed in (vi., 7), where the author virtually writes down without calculation the equation to the five sides of a regular pentagon. Choosing suitable origin and axes this equation in polar coordinates is

$$\Pi_{s=0}^{s=4}[r\cos(\theta-2s\pi/5)-a]=0,$$

or
$$r^5[\cos 5\theta-\cos(5\cos^{-1}a/r)]=0$$

$$r^5(16\cos^5\theta-20\cos^3\theta+5\cos\theta)-16a^5+20a^3r^2-5ar^4=0,$$

or
$$5(x^2+y^2)^2(a-x)-20(x^2+y^2)(a^3-x^3)+16(a^5-x^5)=0.$$

There are very few misprints or inaccuracies. Question (lxxxviii., 6) should read: "The pedal line of any point P on the circum-circle of the triangle ABC cuts the sides at distances x, y, z from the circum-centre. Prove that $\Sigma x^2 . \sin 2A$ is constant" (Bell & Abbott's Scholarship).* This is a curious property the geometrical meaning of which is not apparent.

For those who are desirous of following the trend and the scope of present day scholarship papers there could be no better guide than the present collection. The many things hard to be understood will doubtless prove humbling to pride. But if professional dexterity in problem-solving has to be acquired (and when acquired, maintained) what more invigorating training in this direction could be found than that which is afforded by a persistent and resolute struggle with such veritable Hackenschmidts as these? R. F. Davis.

Theory of Observations. By T. N. Thiele, Directory of the Copenhagen Observatory. London, C. and E. Layton (56 Farringdon Street). Pp. 144+vi. 4to. 1903. 12s.

This work on the Theory of Errors and Method of least Squares can be heartily commended. The treatment is not in the least of the narrow, detailed, and arithmetical kind one might expect; the broad and philosophic mind of the author is reflected on every page, and the discussion on the basis of the theory will be found interesting by many who have never been attracted by its crystallisation into practice.

Teachers who are accustomed to include the elements of the theory of probability in their algebra course will find a welcome stimulus in Professor Thiele's book. Most of it can be read in an arm-chair. E. T. Whittaker.

A Treatise on Hydromechanics. Part I. **Hydrostatics.** By W. H. Besant and A. S. Ramsey. Sixth Edition. Pp. 264. 1904. (Bell.)

Dr. Besant's *Hydrostatics* is too well known to University men to require much more than a passing reference, and congratulations to the author on seeing the sixth edition of this work. But all things change, and it is rare to find anything stereotyped in Mathematics for long. The influence of the doctrine of energetics has been recognised by the author and his collaborator. The chapter on the stability of equilibrium of floating bodies has been enlarged, and the subject is also treated from the point of view of the principle of energy. So, also, the laws of capillarity have been deduced by the same principle from the assumption of the existence of forces between the molecules. Many familiar examples from the

* Math. Trip. 1899 [W. J. G.].

sixties and seventies have disappeared from the new edition, and their places have been filled by others set at recent examinations. We may venture to express regret that Dr. Besant has not seized the opportunity of including in the book historical notes, which would be of great interest to the student. The practical value of the book would have also been enhanced had descriptions been given in some detail of the best known instruments, and it certainly would have made the book more attractive to the average student.

Scientia. Exposé et Développement des questions scientifiques à l'ordre du jour. Published under the direction of MM. APPELL, D'ARSONVAL HALLER, LIPPMANN, MOISSAN, POINCARÉ, MAREY, etc. 2 frcs. each volume. 1903—1904. (Gauthier-Villars.)

This excellent little series, originally published by M. Naud, has now been taken in hand by Messrs. Gauthier-Villars, from whom we have received some half-dozen sample copies. One of these very handy and useful books has already been reviewed in these columns—Laurent's *Elimination*—and Hadamard's *Essay on Taylor's Series* was entrusted by a too confiding editor to a reviewer some two years ago, and that reviewer's conscience will, we hope, experience a sharp prick should his eye meet these lines. The volumes are monographs of about 80-100 pages each. *Géométrographie*, or the art of geometrical constructions, is dealt with by that past master in the art, M. Lemoine, whose name is so intimately connected with the somewhat futile but very ingenious researches of some fifteen years into what was called the *Geometry of the Triangle*. To it we hope to recur. *Diagrammes et Surfaces thermo-dynamiques* is a French translation of the first two memoirs written by the late Willard Gibbs on a subject which is closely connected with the growth of the study of the phenomena of equilibrium in physical and chemical systems. By the application of the dissipation principle, Gibbs was able to construct his theory of systems maintained at a uniform and constant temperature by the gain or loss of external heat—systems known as non-isolated but isothermal. Of his work Professor Larmor says: "His monumental memoir made a clean sweep of the subject; and workers in the modern experimental science of experimental chemistry have returned to it again and again to find their empirical principles forecasted in the light of pure theory, and to derive fresh inspiration for new departures." The memoirs were for long inaccessible to the ordinary student. Ostwald published, in 1891, a German translation of the three classical memoirs which had appeared, in 1873-1878, in the Transactions of the Connecticut Academy, where for long they remained in obscurity. Le Châtelier published a French translation of the essential part of the third memoir in 1899, and this monograph in *Scientia* contains a translation of the first two memoirs. It will familiarise the reader with the ideas of Willard Gibbs on the geometrical representation of thermo-dynamical phenomena by diagrams and surfaces. We hope to have this reviewed more in detail at some future period. The other volumes are not entirely within the purview of the Gazette. M. Cotton writes on the Zeeman Effect. A bright double line is produced if a sodium flame is placed in front of the slit of a spectroscope. But if the flame is placed between the poles of a powerful magnet these lines are found to be broadened and separated into three components. These components are polarised in directions at right angles to each other, the middle line in one plane and the two outer in another. The researches of Lorentz and many others are given in detail by M. Cotton, and he gives an account of Michelson's investigations by the interference method, and of his wonderful echelon spectroscope, which has a much greater resolving power than the grating spectroscope. Zeeman's observation of the effect of magnetism on the rays of light is of great interest to Englishmen, inasmuch as Faraday sought for it in vain "at the time Gauss was in search of the keystone of electro-dynamics" (*v.* that valuable and remarkable book, Merz's *History of European Thought in the Nineteenth Century*, Vol. ii., p. 197. Blackwood.) M. Mendelssohn deals with *Electric Phenomena in Living Beings*; M. Imbert with the *Methods of Economic Functional Activity of the Organism*. The last of the set received is by Le Dantec, and is entitled *La Sexualité*. The author is one of the most daring of modern writers on the mysteries of life and death, and this monograph should be read for its brilliant speculations on some of the most absorbing problems in general biology. We should not under ordinary circumstances venture to draw attention here to books dealing with these subjects. But there must be many among our

younger readers who rest under some disadvantage in their desire to become acquainted with the trend of modern thought in the regions of physics and biology; and we think that in this series they will find the pabulum for which they crave, readily accessible, and at a most moderate cost.

Exercises Méthodiques de Calcul Integral. By E. BRAHY. Pp. viii, 302. 1903. (Gauthier-Villars.)

The private student will find this excellent collection of examples of considerable value, if read and worked out *pari passu* with the usual text-book. The first ten chapters take the reader up to volumes and surfaces. The next eight deal with elementary differential equations. A goodly number of the integrals and equations are worked out at length, and to each section and chapter are appended plenty of worked exercises. The fact that new editions of the exercises in the Differential Calculus, by the author of this little volume, and of Frenet's somewhat similar collection of exercises on the Infinitesimal Calculus, have appeared within the last couple of years would seem to shew that books of this class appeal to a large number of students abroad. In view of the stress that is to be laid in future in the Inter. Sci. and B.Sc. on the knowledge of French and German, this would form a useful volume for those who wish to make themselves acquainted in the former language with the various technical expressions of the Calculus.

Obras sobre Mathematica. By F. GOMES TEXEIRA. Vol. i., pp. 402. 1904. (The University Press, Coimbra.)

This handsome volume, published by Professor Texeira at the instance of the Portuguese Government, contains the first instalment of the many memoirs and notes of the author, which until now have been scattered throughout the pages of mathematical journals in French, Spanish, Portuguese, Italian, and German, the greater number being in the first-named language. They deal with both analytical and geometrical subjects, and include what we believe is the best piece of work the author has done—his researches on the development of functions in series, which have appeared at intervals during the last seven or eight years. Other interesting articles treat of curves parallel to the ellipse, the equipotential curve, Descartes' Ovals, Pascal's Limaçon, and the convergence of formulae of interpolation. Many indeed are the claims on the Treasury of this country, but in our wildest dreams we have never thought of asking the Government of the day to publish even the collected works of a Cayley, a Sylvester, or a Stokes.

Scherz und Ernst in der Mathematik. By W. AHRENS. **Geflügelte und Ungeflügelte Wörte.** Pp. x, 522. 1904. (Teubner.)

This entertaining volume consists of extracts, winged and otherwise, from the writings, addresses, letters, etc., of the most famous German, French, and English mathematicians. A few examples will give an idea of the miscellaneous nature of the contents. "Geometry is the only science which has produced no sects" (Frederick the Great). "D'Alembert, that great genius, seems to be far too ready to pull down everything he has not himself built up" (Euler to Lagrange). "I am delighted at the contrast between your modesty and the good opinion that other geometers have of themselves, although they have certainly nothing like the same claim. You are a living instance of what you said to me some time ago, that pretensions are ever in an inverse ratio to merit" (d'Alembert to Lagrange). "Leibnitz never married. At the age of 50 he began to think about it, but the lady asked for time to reflect. This gave Leibnitz time to reflect—and he did not marry" (Fontenelle). "My dear and illustrious friend—they write to me from Berlin that you are about to take what we philosophers call 'le saut perilleux,' and that you have married one of your relations. . . . Accept my compliments, for a mathematician ought to have pre-eminent advantages in the calculations of—his own happiness, and any calculations of yours are sure to lead to a solution—the solution in your case being marriage" (d'Alembert to Lagrange).

"Petit Poisson deviendra grand
Pourvu que Dieu lui prête vie" (Laplace of S. D. Poisson).

"Sophie Germain's conversation had all the elegance of one of Laplace's beautiful formulae" (Fayolle). The determination of the present Ministry in France to have none but true republicans in the Army reminds us that history repeats itself. It is interesting to recall that the taking of the oath to Napoleon III. was dispensed with in the case of only two men—Arago and Cauchy. Arago had views on the subject, as the following dialogue will shew:

The Minister (*X.*): Now, sir! are you really one of us?

Fresnel: No one is more devoted than I am to our august Royal House, and to the learned institution which France owes to that House.

X.: Yes! but that is too vague. Suppose you were a deputy, which section of the deputies would you join?

Fresnel: I would join Camille Jordan, if he would have me.

X.: I am much obliged for your frankness.

(Next day an unknown gets the post for which Fresnel was applying.)

"My friend! let us take a pinch of snuff; to-morrow I shall not enjoy that pleasure, for my hands will be fastened behind my back" (Bailly to his nephew, the night before his execution). And so on every page. We cannot refrain from quoting in conclusion Hermite's epitaph on Cauchy:

Venerandae memoriae
Eximii viri Augustini Cauchy
Hoc qualecunque munus
Libenter accipias.
Et, ex beata aeternae felicitatis sede
Animum sanctae amicitiae pie memorem
Benigne aspicere digneris.

Lehrbuch der Differential- und Integral-Rechnung. By J. A. SERRET. Edited by G. BOHLMANN and E. ZERMELO. Vol. II. Conclusion. Pp. xii, 305-479. 1904. (Teubner.)

This is a translation of Chapters XI. and XII. of Serret's well-known book. It contains an appendix by Axel Harnack on the integration of partial differential equations in the theory of functions of a complex variable.

Elementare Algebra. Akademische Vorlesungen für Studierende der ersten Semester. By E. NETTO. Pp. viii, 200. 1904. (Teubner.)

This would be found an excellent little manual for the scholarship candidate who requires revision of the algebraical part of his course. It deals, and on general lines, with equations of the first, second, third, fourth, and nth degrees, the binomial and polynomial theorems, combinations, and determinants with their applications to linear equations with more than one unknown. Incidentally there is a good deal of advanced work in these chapters. For example, in the chapter on the general equation of the second degree there are questions on the probability of real roots, or that the roots are of the same sign. Graphical illustrations appear where required.

Practical Geometry for Beginners. By V. LE NEVE FOSTER and F. W. DOBBS. Pp. viii, 96. 1904. (Macmillan.)

This is a good book and has its points. The subject matter is that of the first book of Euclid, "for it seems wiser to avoid trespassing on later principles, and thereby spoiling the student by a premature acquaintance of the joys of subsequent discoveries." The second part is intended to be used concurrently with a course of theory, cross references being appended for those who are reading Mr. Allcock's *Theoretical Geometry for Beginners*.

There are plenty of miscellaneous examples, and patterns are introduced in order to stimulate the pupil to invent designs.

NOTE.

In a notice of Professor Bellino Carrara's "Trisezione dell' Angolo" (Vol. iii., p. 65) the writer states that there is no mention in this *brochure* of Klein's well-known *Vorträge über ausgewählte Fragen der Elementargeometrie*. He now begs to make the *amende honorable* to Professor Carrara. It is mentioned on p. 17.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. HENDRICKS, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

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The Messenger of Mathematics. Vols. 24, 25.

Tortolini's Annali. Vol. I. (1850), or any of the first eight parts of the volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

BOOKS, ETC., RECEIVED.

On the General Equations of Mechanics. By P. E. B. JOURDAIN. Pp. 61-79. Quart. Jour. Math. No. 141. 1904.

The Oxford and Cambridge Graphical Algebra. By R. H. ALLPRESS and F. MARSHALL. Pp. 26. 6d. 1904. (Gill.)

Woher kommen die Weltgesetze? By J. KÜBLER. Pp. 30. 1 m. 1904. (Teubner.)

Elementary Trigonometry. By C. H. P. MAYO. Pp. x, 204. 3s. 6d. 1904. (Longmans.)

An Introduction to the Modern Theory of Equations. By FLORIAN CAJORI. Pp. x, 233. 7s. 6d. net. 1904. (Macmillan.)

Encyclopédie des Sciences Mathématiques Pures et Appliquées. Tome I. Vol. I. Fascicule I. Edited by PROFESSOR JULES MOLK and F. MEYER. French Edition. Pp. 160. 5 fr. or 4 mk. 1904. (Gauthier-Villars or Teubner.)

American Journal of Mathematics. Edited by FRANK MORLEY. Vol. XXVI. No. 4. Oct., 1904. (Kegan Paul.)

Invariants of Linear Partial Differential Equations and the Theory of Congruences of Rays. E. J. WILCZYNSKI. *On Elements connected each to each by one or the other of Two Reciprocal Relations.* C. DE POLIGNAC.

The Lobachevski Prize. By G. B. HALSTED. (*Science*, Sept. 16.)

Über eine zeitgemässe Umgestaltung der mathematischen Unterrichts an der höheren Schulen. By F. KLEIN. Pp. 82. 1 m. 60. 1904. (Teubner.)

Beiträge zur Frage des Unterrichts in Physik und Astronomie an den höheren Schulen. By E. REICKE. Pp. 190. 2 m. 1904. (Teubner.)

Methodisches Lehrbuch der Elementarmathematik. By G. HOLZMULLER. Vol. I. Pp. 317. 2 m. 80. 1904. (Teubner.)

Scherz und Ernst in Mathematik. By W. AHRENS. Pp. 522. 8 m. 1904. (Teubner.)

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